Petermichl's Dyadic Shift

Hilbert transform: $Hf(x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy$ (1905, D. Hilbert)

$$= \text{p.v. } \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy$$

(define first for Schwartz functions, extend to $L^2(\mathbb{R})$ etc.)

The Hilbert transform kernel $K_H(x,t) := k_H(x-t) = \frac{1}{x-t}$

Characterizing Properties: Let T be a bounded operator on $L^2(\mathbb{R})$ that:
- commutes with translations & dilations;
- anticommutes w/ reflections: $(Tf)^{\sim} = -T(\tilde{f})$ $\qquad \tilde{f}(x) := f(-x)$

Then T is a constant multiple of the Hilbert transform H: $T = cH$, for some $c \in \mathbb{R}$.

Random Dyadic Grids (original S.P. formulation)

→ Recall the standard dyadic grid $\mathcal{D}^0$ on $\mathbb{R}$, consisting of all intervals of the form

$$\mathcal{D}^0 := \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) : k, n \in \mathbb{Z} \right\}$$

and its associated Haar system $\{h_I\}_{I \in \mathcal{D}^0}$, $h_I = \frac{1}{\sqrt{|I|}}(\mathbb{1}_{I_+} - \mathbb{1}_{I_-})$.

→ Obtain variations of $\mathcal{D}^0$ by:
- → Shifting the "starting point" 0 to $\alpha \in \mathbb{R}$ $\quad ([a,b) \mapsto [a+\alpha, b+\alpha))$
- → Scaling the lengths by some $r > 0$: $\quad ([a,b) \mapsto [ra, rb))$

The resulting grid: $\mathcal{D}^{\alpha,r}$ → Satisfies → partitioning: each generation (length $r2^n$) partitions $\mathbb{R}$

→ the nested, one-parent & two equal-length children per interval properties

(loses the 2^n-length property, but preserves the essential geometrical properties).

→ The Haar system adapted to $\mathcal{D}^{\alpha,r}$: $\{h_I^{\alpha,r}\}_{I \in \mathcal{D}^{\alpha,r}}$ are defined in the same way: $\left[\frac{1}{\sqrt{|I|}}(\mathbb{1}_{I_+} - \mathbb{1}_{I_-})\right]$ and preserve the properties of usual Haar functions (onb for $L^2(\mathbb{R})$, etc.)

$$f = \sum_{I \in \mathcal{D}^{\alpha,r}} (f, h_I)\, h_I \quad , \forall f \in L^2(\mathbb{R}).$$

→ The dyadic shift $\text{Ш}^{\alpha,r}$ corresponding to the dyadic grid $\mathcal{D}^{\alpha,r}$ is defined by:

$$\boxed{\text{Ш}^{\alpha,r} f(x) := \sum_{I\in\mathcal{D}^{\alpha,r}} (f,h_I)\left(h_{I_+}(x) - h_{I_-}(x)\right)}$$

→ Bounded operator on $L^2(\mathbb{R})$, with norm $\sqrt{2}$:

$$\|\text{Ш}^{\alpha,r} f\|_{L^2}^2 = \sum_{I\in\mathcal{D}^{\alpha,r}} \left(\text{Ш}^{\alpha,r} f, h_I\right)^2 = \sum_{I\in\mathcal{D}^{\alpha,r}} 2(f,h_I)^2 = 2\|f\|^2_{L^2(\mathbb{R})}$$

→ Representing Kernel:

$$(\text{Ш}^{\alpha,r} f)(x) = \int_{\mathbb{R}} K^{\alpha,r}(t,x) f(t)\,dt \;;\quad K^{\alpha,r}(t,x) := \sum_{I\in\mathcal{D}^{\alpha,r}} h_I(t)\left(h_{I_+}(x) - h_{I_-}(x)\right)$$

$$= \int_{\mathbb{R}} h_I(t) f(t) \left(h_{I_+}(x) - h_{I_-}(x)\right) dt$$

→ **Main Idea**: The individual dyadic shifts $\text{Ш}^{\alpha,r}$ do not commute with translations & dilations, nor do they anticommute w/ reflections. However an average over all dyadic grids $\mathcal{D}^{\alpha,r}$ does.

→ Example: Look at antisymmetry, i.e. how Ш interacts with reflections:

Let $I\in\mathcal{D}^{\alpha,r}$, so I is of the form $I=[r(\alpha+a), r(\alpha+b))$ for some $[a,b)\in\mathcal{D}^0$.

Remark: $\tilde{I} := [r(-\alpha-b), r(-\alpha-a))$ belongs to $\mathcal{D}^{-\alpha,r}$ and has $|\tilde{I}| = |I|$.

Moreover: $\left.\begin{array}{l} \mathbb{1}_{I_+}(-t) = \mathbb{1}_{\tilde{I}_-}(t) \\ \mathbb{1}_{I_-}(-t) = \mathbb{1}_{\tilde{I}_+}(t) \end{array}\right\} \Rightarrow \boxed{h_I^{\alpha,r}(-t) = -h_{\tilde{I}}^{-\alpha,r}(t)}$

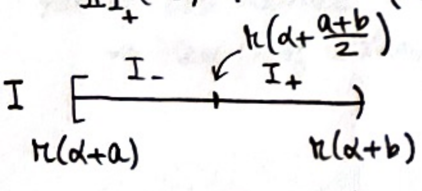
$\mathbb{1}_{I_+}(-t) = 1 \iff r(\alpha + \frac{a+b}{2}) \leq -t < r(\alpha+b)$

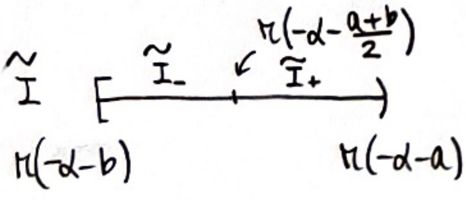
$\Updownarrow$

$r(-\alpha - \frac{a+b}{2}) \geq t > r(-\alpha-b)$

i.e. $t\in\tilde{I}_-$

$$h_I^{\alpha,r}(-t) = \frac{1}{\sqrt{|I|}}\left(\mathbb{1}_{I_+}(-t) - \mathbb{1}_{I_-}(-t)\right) = \frac{1}{\sqrt{|\tilde{I}|}}\left(\mathbb{1}_{\tilde{I}_-}(t) - \mathbb{1}_{\tilde{I}_+}(t)\right) = -h_{\tilde{I}}^{-\alpha,r}(t).$$





$$\Rightarrow \boxed{K^{\alpha,r}(-t,-x) = -K^{-\alpha,r}(t,x)}$$

→ Similar relationships w/ translations: $K^{\alpha,r}(t+c, x+c) = K^{\alpha-c,r}(t,x)$

& dilations: $K^{\alpha,r}(\lambda t, \lambda x) = K^{\alpha/\lambda, r/\lambda}(t,x)\frac{1}{\lambda}.$

→ In "Dyadic shifts & a logarithmic estimate for Hankel operators w/ matrix symbol" (C.R. Acad. Sci. Paris, 2000) Stefanie Petermichl showed that

$$K(t,x) := \lim_{L\to\infty} \frac{1}{2\log L} \int_{1/L}^{L} \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} K^{\alpha,r}(t,x)\, d\alpha \,\frac{dr}{r}$$

is well-defined and satisfies

- Translation invariance: $K(t,x) = K(t+c, x+c)$ $\quad c\in\mathbb{R}$
- Dilation invariance: $K(t,x) = \lambda K(\lambda t, \lambda x)$ $\quad \lambda>0$
- Antisymmetry: $\boxed{K(t,x) = -K(-t,-x)}$

for example, this uses: $K^{\alpha,r}(-t,-x) = -K^{-\alpha,r}(t,x)$

$$\Rightarrow K(-t,-x) = \lim_{L\to\infty} \frac{1}{2\log L} \int_{1/L}^{L} \lim_{R\to\infty} \frac{1}{2R} \underbrace{\int_{-R}^{R} K^{\alpha,r}(-t,-x)\, d\alpha\, \frac{dr}{r}}_{\int_{-R}^{R} -K^{-\alpha,r}(t,x)\, d\alpha\, \frac{dr}{r}}$$

Change of var: $a = -\alpha$

$$= \int_{R}^{-R} -K^{a,r}(t,x)(-da)\frac{dr}{r}$$

$$= -\int_{-R}^{R} K^{\alpha,r}(t,x)\, d\alpha\, \frac{dr}{r}$$

$$\Rightarrow K(-t,-x) = -\lim_{L\to\infty} \frac{1}{2\log L} \int_{1/L}^{L} \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} K^{-\alpha,r}(t,x)\, d\alpha\, \frac{dr}{r}$$

$$= -K(t,x).$$

⇒ These properties show then that $K(t,x) = \frac{C_0}{t-x}$ is a multiple of the Hilbert Kernel.

→ More proof is then required to see that $C_0 \neq 0$.

Usage Example: Commutators.

To control $[b,H]f = b\cdot Hf - H(bf)$, it suffices to control $[b, \text{Ш}]$, with Ш: bounds independent from the choice of dyadic grid.

Split into paraproducts:

$$
\begin{aligned}
[b,\text{Ш}]f &= b\cdot \text{Ш}f - \text{Ш}(bf) \\
&= \Pi_b \text{Ш}f + \Pi_b^* \text{Ш}f + \Pi_{\text{Ш}f}\, b - \text{Ш}\left(\Pi_b f + \Pi_b^* f + \Pi_f b\right) \\
&= \underbrace{\left(\Pi_b \text{Ш} + \Pi_b^* \text{Ш} - \text{Ш}\Pi_b - \text{Ш}\Pi_b^*\right)}_{\text{use known individual bounds for each operator (w.r.t. weights involved)}} f + \underbrace{\left(\Pi_{\text{Ш}f}\, b - \text{Ш}\Pi_f b\right)}_{Rf \text{ remainder term}}
\end{aligned}
$$

<u>Remark</u>: Really just $[\Pi_b, \text{Ш}] + [\Pi_b^*, \text{Ш}]$

one needs to use cancellation that (hopefully) occurs to bound this.

$$\text{Ш}f(x) = \sum_{I\in\mathcal{D}} (f, h_I)\left(h_{I_+}(x) - h_{I_-}(x)\right)$$

$$(\text{Ш}f, h_J) = \left(\sum_I (f,h_I)(h_{I_+} - h_{I_-}), h_J\right) = \begin{cases} (f, h_{\hat{J}}) & \text{if } J = \hat{J}_+ \\ -(f, h_{\hat{J}}) & \text{if } J = \hat{J}_- \end{cases}$$

$J = I_+$ or I_-

$\Rightarrow I = \hat{J}$

$$\Rightarrow \Pi_{\text{Ш}f}\, b = \sum_J (\text{Ш}f, h_J)\langle b\rangle_J h_J = \sum_{\hat{J}} (f, h_{\hat{J}})\left(\langle b\rangle_{\hat{J}_+} h_{\hat{J}_+} - \langle b\rangle_{\hat{J}_-} h_{\hat{J}_-}\right)$$

$$= \sum_J (f, h_J)\left(\langle b\rangle_{J_+} h_{J_+} - \langle b\rangle_{J_-} h_{J_-}\right)$$

$$\text{Ш}\Pi_f b = \sum_J (\Pi_f b, h_J)(h_{J_+} - h_{J_-}) = \sum_J (f, h_J)\langle b\rangle_J (h_{J_+} - h_{J_-})$$

$$\Rightarrow R f = \Pi_{Ш f} b - Ш \Pi_f b$$

$$\frac{1}{2}\left(\langle b\rangle_{J_-} + \langle b\rangle_{J_+}\right)$$

$$= \sum_J (f, h_J)\left[\left(\langle b\rangle_{J_+} - \langle b\rangle_J\right) h_{J_+} - \left(\langle b\rangle_{J_-} - \langle b\rangle_J\right) h_{J_-}\right]$$

$$= \sum_J (f, h_J)\left[\frac{1}{2}\underbrace{\left(\langle b\rangle_{J_+} - \langle b\rangle_{J_-}\right)}_{\frac{2}{\sqrt{|J|}}(b, h_J)} h_{J_+} - \frac{1}{2}\underbrace{\left(\langle b\rangle_{J_-} - \langle b\rangle_{J_+}\right)}_{-\frac{2}{\sqrt{|J|}}(b, h_J)} h_{J_-}\right]$$

$$(b, h_J) = \frac{1}{\sqrt{|J|}}\left(\int_{J_+} b - \int_{J_-} b\right) = \frac{\sqrt{|J|}}{2}\left(\langle b\rangle_{J_+} - \langle b\rangle_{J_-}\right)$$

$$= \sum_J (f, h_J)\frac{1}{\sqrt{|J|}}(b, h_J)\left(h_{J_+} + h_{J_-}\right)$$

So, for example, if we are showing boundedness of $[b, H]$ in $L^2(w)$, $w \in A_2$:
Take $f \in L^2(w)$, $g \in L^2(w^{-1})$:

$$(Rf, g) = \sum_J (f, h_J)(b, h_J)\frac{1}{\sqrt{|J|}}\left((g, h_{J_+}) + (g, h_{J_-})\right)$$

$$= \Big(b, \underbrace{\sum_J (f, h_J)\frac{1}{\sqrt{|J|}}\left((g, h_{J_+}) + (g, h_{J_-})\right) h_J}_{\phi}\Big)$$

$$\lesssim \|b\|_{BMO}\|\phi\|_{\mathcal{H}^1}$$

$$|(\varphi, h_K)| = \frac{1}{\sqrt{|K|}}\left|\int_{K_+}\varphi - \int_{K_-}\varphi\right| \leq \frac{1}{\sqrt{|K|}}\int_K |\varphi|$$

$$\boxed{|(\varphi, h_K)| \leq \sqrt{|K|}\,\langle|\varphi|\rangle_K}$$

$$S^2\phi = \sum_J (f, h_J)^2 \frac{1}{|J|}\underbrace{\left((g, h_{J_+}) + (g, h_{J_-})\right)^2}\frac{\mathbb{1}_J}{|J|}$$

$$\leq \left(|(g, h_{J_+})| + |(g, h_{J_-})|\right)^2$$

$$\leq \left(\sqrt{\frac{|J|}{2}}\right)^2\left(\langle|g|\rangle_{J_+} + \langle|g|\rangle_{J_-}\right)^2$$

$$= \frac{|J|}{2}\cdot\left(2\langle|g|\rangle_J\right)^2 = 2|J|\langle|g|\rangle_J^2$$

$$\leq 2\sum_J (f, h_J)^2\langle|g|\rangle_J^2\frac{\mathbb{1}_J}{|J|} \leq 2(Mg)^2(Sf)^2$$

$$\Rightarrow \|\phi\|_{\mathcal{H}_1} \lesssim \int (Mg)(Sf)\,dx \leq \|Mg\|_{L^2(w^{-1})}\|Sf\|_{L^2(w)} \lesssim \|g\|_{L^2(w^{-1})}\|f\|_{L^2(w)}.$$